Polynomial Wolff axioms and Kakeya-type estimates for bent tubes

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Kakeya Conjecture

Let $\mathbb{T}$ be a set of tubes in $\mathbb{R}^n$ pointing in δ-separated directions. Assume this set to be maximal size, i.e. $|\mathbb{T}| \sim \delta^{1-n}$. Then up a loss factor of $\delta^{-\epsilon}$, the volume of the union of these tubes is roughly one, that is

$$\left| \bigcup_{T \in \mathbb{T}} T \right| \geq C_\epsilon \delta^\epsilon$$

Notation

Definition

Let A and B be (take on) real numbers. Writing $A \lesssim B$ means there exists a constant C such that $A \leq CB$. Writing $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

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Let A and B be real numbers. Writing $A \lessapprox B$ means for all $\epsilon > 0$, there exists a constant C_ϵ such that $A \leq C_\epsilon \delta^{-\epsilon} B$. Similarly, $A \approx B$ denotes that $A \lessapprox B$ and $B \lessapprox A$.

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- You can break it up into multilinear pieces
- Assume tubes point in k, $k = 1, \ldots, n$, linearly independent directions
- 1-linear
  - Use induction on scales
- n-linear
  - Bennett-Carbery-Tao
- 2-linear and 3-linear are harder
  - Naive approach: Add extra direction and apply 4-linear

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Wolff vs. Polynomial Wolff

- We add assumptions about the tubes

Linear Wolff

Not too many tubes lie within a hyperplane. That is at most t/δ tubes lie within the rectangular prism of dimensions $1 \times t \times \delta \times \cdots \times \delta$.

Polynomial Wolff

Not too many tubes lie within the neighborhood of an algebraic variety. That is for any semi-algebraic set of complexity E there is a constant K_E such that for all $\lambda \geq \delta$

$$\#\{T \in \mathbb{T} : |T \cap S| \geq \lambda |T|\} \leq K_E |S| \delta^{1-n} \lambda^{-n}$$

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Current Work

- Guth and Zahl's trilinear bounds are amiable to certain bent tubes

Definition

A bent δ-tube is an algebraic curve of degree $\lesssim 1$ and C^2-norm $\lesssim 1$.

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Trilinear Bound

Let $\mathbb{T}$ be a set of bent δ-tubes in $\mathbb{R}^4$ that satisfy the polynomial Wolff axioms, then

$$\int \Big(\sum_{T_1, T_2, T_3 \in \mathbb{T}} \chi_{T_1} \chi_{T_2} \chi_{T_3} |v_1 \wedge v_2 \wedge v_3|^{12/13} \Big)^{13/27} \lessapprox K^{1/9} \delta^{-1/3} (\delta^3 |\mathbb{T}|)^{4/3}$$

where $K = \max_{1 \le E \le d(\epsilon)} K_E$.

- $v_i(x)$ is the unit tangent vector to T_i at x
- $\chi_{T_i}(x) = \begin{cases} 1 & x \in T_i \\ 0 & \text{otherwise} \end{cases}$

Remark: Choosing ϵ smaller and reducing the error $\delta^{-\epsilon}$ requires use of higher degree algebraic varieties.

Future Work

- Consider broader categories of bent tubes

Definition

A bent δ-tube is the δ-neighborhood of a length C^k-curve with C^k-norm $\lesssim 1$.

Modified Polynomial Wolff Axiom

Let Ω be a C^k surface with C^k-norm $\leq C$, then there exists a constant K_C such that for all $\lambda \geq \delta$ and $r > 0$ we have

$$\#\{T : |T \cap N_r(\Omega)| \geq \lambda |T|\} \leq K_C |N_r(\Omega)| \delta^{1-n} \lambda^{-n}$$

where $N_r(\Omega)$ is the r-neighborhood of Ω.

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